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Kosterlitz–Thouless Transition for the Finite-Temperature $d = 2 + 1$, $U(1)$ Hamiltonian Lattice Gauge Theory

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We prove that in the $d = 2 + 1$, $U(1)$ Hamiltonian (continuous time) lattice gauge theory the confining potential between two static external charges grows logarithmically with their distance, at sufficiently high temperatures. As it is known that for zero or low temperatures and large coupling constant the model confines linearly, we have therefore established the existence of a Kosterlitz–Thouless transition. Our results are based on a Mermin–Wagner type of argument combined with correlation inequalities and known results for the two-dimensional (spin) Villain model.

KEY WORDS: Kosterlitz–Thouless phase transition; finite-temperature gauge theory; Hamiltonian lattice gauge theory.

1. INTRODUCTION

It is by now a well-known fact that the interaction potential of a static "quark–antiquark" pair in a 3-dimensional $U(1)$ lattice gauge theory is confining. The first rigorous result was probably obtained by Glimm and Jaffe.(1) They used a technique developed by McBryan and Spencer(2) to prove that the interaction potential increases at least logarithmically with the distance. Their result applied to a discrete "time" version of the model and at zero temperature for all values of the coupling. Stronger results which give linearly increasing potential at zero temperature were obtained by Göpfert and Mack.(3) More recently, Borgs(4) showed that linear confinement holds also for low temperatures and strong coupling. His

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result remains true in the quantum version (continuous "time") of the $U(1)$ model.

The reason why the 3-dimensional $U(1)$ model exhibits permanent confinement of static "quarks" is related to the Mermin–Wagner phenomenon,[16,5] that is, the absence of spontaneous breaking of a continuous symmetry at low dimensions. This mechanism implies *a priori* bounds for the decay rate of certain correlation functions which are related to the interaction potential.[5,6]

In this paper we consider a two-dimensional lattice Hamiltonian gauge theory, with symmetry group $U(1)$. Such a theory is equivalent to a three-dimensional $U(1)$ Euclidean lattice gauge theory in the limit of zero lattice spacing in "time" direction (see, for instance, refs. 7–9 and references therein). We first present a quantum version of the Spencer–McBryan ideas[2] to show that the basic Glimm–Jaffe result survives the continuum time limit, by proving that the interaction potential is at least logarithmically increasing with the distance for all values of the coupling g. The result is proved to be true at temperatures $\beta^{-1} \geqslant 0$ and applies also to any symmetry group whose center contains $U(1)$.[10,11] Further, by the use of correlation inequalities we show that for high enough values of $(\beta g)^{-1}$ the interaction potential is bounded from above by a logarithmic function. Hence, we conclude that at high temperatures (or small coupling) the long-distance asymptotic behavior of the potential *is* logarithmic. Combined with the results in ref. 4, we have therefore rigorously established the existence of a Kosterlitz–Thouless type of transition, with the potential changing from linear to logarithmic at a critical temperature $T_c(g)$ for g large enough. Our findings confirm the heuristic arguments of ref. 15.

In Section 2 we define the model and state the confinement problem. The main results and proofs are given in Section 3.

2. THE MODEL AND THE CONFINEMENT PROBLEM

We shall work in the square lattice $\mathbb{Z}^2$ (unit spacing).

To have things well defined, we will set the model in a finite square volume $\Lambda \subset \mathbb{Z}^2$ (with periodic boundary conditions for instance) and we take $\Lambda \to \mathbb{Z}^2$ at the end. In order to have the notation as simple as possible, we will not write explicitly the volume dependence in most of the following formulas.

A unit vector in the k $(k = 1, 2)$ coordinate direction will be denoted by $\hat{k}$, while $\mathbb{Z}^{2*}$ denotes the set of ordered links in $\mathbb{Z}^2$,

$$\mathbb{Z}^{2*} = \{l \equiv (x, x + \hat{k}), x \in \mathbb{Z}^2, k = 1, 2\}$$

To each link $l=(x, x+\hat{k})$ there corresponds a copy of the Hilbert space $L^2[0, 2\pi]$, link variables $\theta_l=\theta(x, k)$; $0 \leqslant \theta_l < 2\pi$, and angular momentum operators in $L^2[0, 2\pi]$,

$$L_l \equiv L(x, k) = \frac{1}{i} \frac{\partial}{\partial \theta_l}$$

with periodic boundary conditions.

The Hilbert space for the model is the tensor product $\mathscr{H} = \otimes_{x,k} L^2[0, 2\pi]$. The Hamiltonian operator for coupling constants $g_E > 0$, g_M is given by

$$H = -g_E \sum_l \frac{\partial^2}{\partial \theta_l^2} - g_M \sum_{\substack{x \\ k>j}} \cos[\theta(x, k) + \theta(x+\hat{k}, j) - \theta(x+\hat{j}, k) - \theta(x, j)]$$

We will often write for short

$$H = g_E H_E + g_M H_M = g_E \sum_l L_l^2 - g_M \sum_p \cos \theta_p$$

where

$$\theta_p = \sum_{l \in p} \theta_l = \theta(x, k) + \theta(x+\hat{k}, j) - \theta(x+\hat{j}, k) - \theta(x, j)$$

with $\theta_l = -\theta_{-l}$ and p denotes an oriented plaquette in $\mathbb{Z}^2$ (see Fig. 1).

We now define gauge transformations: given a function φ from $\mathbb{Z}^2$ to $[0, 2\pi]$, i.e.,

$$\varphi\colon\ x \in \mathbb{Z}^2 \to \varphi(x) \in [0, 2\pi]$$

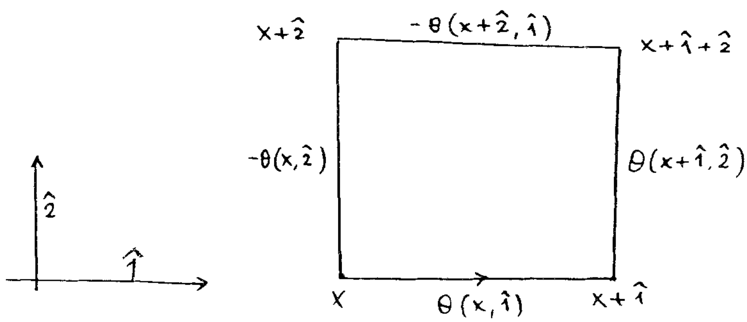

Fig. 1. An oriented plaquette.

they are defined by the map $[0, 2\pi]^{\mathbb{Z}^2} \to [0, 2\pi]^{\mathbb{Z}^2}$,

$$\theta(x, k) \to \theta(x, k) - \varphi(x) + \varphi(x + \hat{k})$$

A gauge transformation $\boldsymbol{\varphi}$ is unitarily implemented by the operator

$$U(\boldsymbol{\varphi}) = \prod_{x,k} \exp\{i[\varphi(x) - \varphi(x + \hat{k})]\, L(x, k)\}$$

Alternatively, $U(\boldsymbol{\varphi})$ can be written

$$U(\boldsymbol{\varphi}) = \exp\left\{i \sum_x \varphi(x)[L(x, k) - L(x - \hat{k}, k)]\right\} \tag{1}$$

Gauge invariance may be stated as

$$U(\boldsymbol{\varphi})\, H U^{-1}(\boldsymbol{\varphi}) = H \tag{2}$$

for all gauge transformations $\boldsymbol{\varphi}$.

It follows from (1) and (2) that

$$[Q(x), H] = 0$$

where

$$Q(x) = \sum_k [L(x, k) - L(x - \hat{k}, k)] \tag{3}$$

$Q(x)$ is the local generator of gauge transformations and by (2) it defines a locally conserved external charge with eigenvalues

$$q(x) = 0, \pm 1, \pm 2, \ldots$$

Therefore we can decompose $\mathscr{H}$ into sectors (subspaces) of well-defined external charges, each of these sectors being invariant by the time evolution operator $\exp(-itH)$. They are labeled by functions

$$\mathbf{q}\colon\ x \in \mathbb{Z}^2 \to q(x) \in \mathbb{Z}$$

which signal the value of the external charge $q(x)$ sitting at $x \in \mathbb{Z}^2$.

The vacuum state (the ground state of H) belongs to the sector with no charges, the one labeled by the function $\mathbf{q} \equiv 0$.[8] The Hilbert space $\mathscr{H}$ allows not only the situation of no external charges, but also any given distribution of charges, $\mathbf{q}$.

We may write the projection operator onto each of these sectors as

$$P_{\mathbf{q}} = \frac{1}{(2\pi)^{|\Lambda|}} \int_0^{2\pi} [\exp(-i\boldsymbol{\varphi} \cdot \mathbf{q})]\, U(\boldsymbol{\varphi})\, d\boldsymbol{\varphi}$$

where $\mathbf{q} \cdot \boldsymbol{\varphi} = \sum_x q(x)\, \varphi(x)$ and $d\boldsymbol{\varphi} \equiv \prod_x d\varphi(x)$.

We are now in a position to define the interaction potential between two external charges q and $-q$ ("quark" and "antiquark" pair) at relative distance R. Owing to the translational invariance, we may put the charge q at $x=0$ and $-q$ at $x=R$. The interaction potential at inverse temperature β is then defined as the difference between the free energy corresponding to the configuration with the two charges minus the free energy of the vacuum sector (no charges).

More precisely, if we define the configuration $\mathbf{q}_2$ by

$$\mathbf{q}_2(x)=\begin{cases} q & \text{for} \quad x=0 \\ -q & \text{for} \quad x=R \\ 0 & \text{otherwise} \end{cases}$$

the interaction potential $V(\beta, R)$ is given by

$$V(\beta, R)=-\frac{1}{\beta}\lim_{\Lambda\to\infty}\ln\left(\frac{Z_{\mathbf{q}_2}}{Z_0}\right) \tag{4}$$

where $Z_{\mathbf{q}_2}=\mathrm{Tr}[\exp(-\beta H)\,P_{\mathbf{q}_2}]$ and $Z_0=\mathrm{Tr}[P_0\exp(-\beta H)]$. Here $P_{\mathbf{q}_2}$ is the projector onto the sector with the two specified charges and P_0 projects onto the vacuum sector. [Implicit in (4) is the infinite-volume limit $\Lambda\to\mathbb{Z}^2$.]

To obtain the interaction potential $V(R)$ at zero temperature we have to take first the limit $\beta\to\infty$ (keeping $|\Lambda|$ fixed) and only then taking $\Lambda\nearrow\mathbb{Z}^2$:

$$V(R)=-\lim_{\Lambda\to\infty}\lim_{\beta\to\infty}\frac{1}{\beta}\ln\left(\frac{Z_{\mathbf{q}_2}}{Z_0}\right)$$

3. RESULTS AND PROOFS

Our first result may be stated as follows.

Theorem 1. The infinite-volume limit interaction potential $V(\beta, R)$ of the two-dimensional $U(1)$ Hamiltonian lattice gauge theory satisfies

$$V(\beta, R)\geqslant g_E(2q-1)\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\frac{1-\cos p\cdot R}{E(p)}$$

for all values of β, $g_E>0$, g_M and where $E(p)=2\sum_{k=1,2}(1-\cos p_k)$ with $p\cdot R=p_1R_1+p_2R_2$.

Remark. The logarithmic lower bound on $V(\beta, R)$ follows from the theorem and from the inequality[5]

$$\int_{-\pi}^{\pi} \frac{1-\cos p\cdot R}{E(p)} \frac{d^2p}{(2\pi)^2} \geqslant \frac{1}{2\pi} \ln |R|$$

Proof. Using Trotter's product formula, we have

$$\begin{aligned} \frac{Z_{\mathbf{q}_2}}{Z_0} &= \frac{1}{Z_0} \lim_{n\to\infty} S_n \\ &= \frac{1}{Z_0} \lim_{n\to\infty} \operatorname{Tr}\left[\exp\left(-\frac{\beta g_E}{n} H_E\right) P_{\mathbf{q}_2} \exp\left(-\frac{\beta g_M}{n} H_M\right)\right]^n \end{aligned}$$

The kernel of the operator $\exp(-\alpha H_E)\, P_{\mathbf{q}_2}$ can be easily computed:

$$\begin{aligned} &[P_{\mathbf{q}_2} \exp(-\alpha H_E)](\boldsymbol{\theta}, \boldsymbol{\theta}') \\ &\quad = \frac{1}{(2\pi)^{|\Lambda|}} \int_0^{2\pi} d\boldsymbol{\varphi} \exp\{-iq[\varphi(0)-\varphi(R)]\} \\ &\qquad \times \prod_{x,k} \left\{ \left(\frac{2\pi}{\alpha}\right)^{1/2} \sum_{m=-\infty}^{+\infty} \exp\left[-\frac{1}{2\alpha}(\theta(x,\hat{k}) - \theta'(x,k) \right.\right. \\ &\qquad \left.\left. + \varphi(x) - \varphi(x+\hat{k}) + 2\pi m)^2 \right]\right\} \end{aligned} \tag{5}$$

From (5) it follows that

$$\begin{aligned} S_n &= \frac{1}{(2\pi)^{n|\Lambda|}} \int_0^{2\pi} \left(\prod_{i=1}^n d\boldsymbol{\theta}_i\, d\boldsymbol{\varphi}_i\right) \exp\left\{-iq \sum_{i=1}^n [\varphi_i(0) - \varphi_i(R)]\right\} \\ &\quad \times \prod_{i=1}^n \left\{ \exp\left[-\frac{\beta g_M}{n} H_M(\boldsymbol{\theta}_{i+1})\right] \right. \\ &\quad \times \prod_{x,k} \left[\left(\frac{2\pi n}{\beta g_E}\right)^{1/2} \sum_{m=-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{n}{\beta g_E}[\theta_i(x,k) - \theta_{i+1}(x,k) \right.\right. \\ &\quad \left.\left.\left. + \varphi_i(x) - \varphi_i(x+\hat{k}) + 2\pi m]^2\right\}\right]\right\} \end{aligned} \tag{6}$$

Next we perform an imaginary translation[2] for each $x \in \Lambda$:

$$\varphi_i(x) \to \varphi_i(x) + i\alpha(x)$$

and estimate the integrand in absolute value to obtain

$$
\begin{aligned}
|S_n| \leqslant & \exp\{-nq[\alpha(0)-\alpha(R)]\} \\
& \times \exp\left\{\frac{1}{2}\frac{n^2}{\beta g_E}\sum_{x,k}[\alpha(x)-\alpha(x+\hat{k})]^2\right\} \\
& \times \operatorname{Tr}\left[\exp\left(-\frac{\beta g_E}{n}H_E\right)\exp\left(-\frac{\beta g_M}{n}H_M\right)P_0\right]^n
\end{aligned}
\tag{7}
$$

We now choose

$$
\alpha(x)=\frac{\beta g_E}{n}c_R(x), \qquad x\in\mathbb{Z}^2
$$

with

$$
c_R(x)=\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{\cos p\cdot x-\cos p\cdot(x-R)}{E(p)}
$$

Here Λ^* stands for the set of allowed values of the "momentum" variable p.

The following relations can be easily verified:

$$
\sum_{x,k}[\alpha(x)-\alpha(x+\hat{k})]^2 \leqslant \frac{\beta^2 g_E^2}{n^2}\frac{2}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{1-\cos p\cdot R}{E(p)}
$$

and

$$
\alpha(0)-\alpha(R)=\frac{\beta g_E}{n}\frac{2}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{1-\cos p\cdot R}{E(p)}
\tag{8}
$$

Inserting (7) and (8) in (6) and taking the limit $n\to\infty$, we get

$$
\frac{Z_{\mathfrak{q}_2}}{Z_0}\leqslant \exp\left\{-\beta g_E(2q-1)\frac{1}{|\Lambda|}\sum_{p\in\Lambda^*}\frac{1-\cos p\cdot R}{E(p)}\right\}
\tag{9}
$$

Finally, taking the logarithm of (9) and the limit $\Lambda \nearrow \mathbb{Z}^2$, we obtain

$$
V(\beta, R)\geqslant g_E(2q-1)\int_{-\pi}^{\pi}\frac{d^2p}{(2\pi)^2}\frac{1-\cos p\cdot R}{E(p)}
\tag{10}
$$

and this ends the proof of the theorem.

Remarks. 1. Obviously, as the estimate (10) does not depend on β, it holds also for $\beta = \infty$ (zero temperature).

2. As pointed out in the Introduction, the reason for the existence of permanent confinement (at least logarithmic) is due to the existence of a global, two-dimensional, $U(1)$ symmetry in addition to the local gauge symmetry. This is manifest in expression (6) by the transformation $\varphi_i(x) \to \varphi_i(x) + \alpha$ (i fixed, $\alpha \in [0, 2\pi]$, $\forall x \in \Lambda$). The Mermin–Wagner mechanism forbids the spontaneous breaking of such a symmetry and imposes[6] an *a priori* decay for correlations of functions of the φ_i. This is at the root of the validity of the bound (10) for all values of β and g.

To proceed further, let us define an expectation value of an observable $A(\boldsymbol{\theta}, \boldsymbol{\varphi})$ by

$$\begin{aligned}\langle A\rangle(\beta g_E, \beta g_M) &= \frac{1}{Z} \lim_{n\to\infty} \int_{-\pi}^{\pi} \left(\prod_{i=1}^{n} d\boldsymbol{\theta}_i\, d\boldsymbol{\varphi}_i \right) A(\boldsymbol{\theta}, \boldsymbol{\varphi}) \\ &\times \prod_{i=1}^{n} \left(\exp\left[-\frac{\beta g_M}{n} H_M(\boldsymbol{\theta}_{i+1}) \right] \prod_{x,k} \left\{ \left(\frac{2\pi n}{\beta g_E} \right)^{1/2} \right.\right. \\ &\left.\left. \times \sum_{m=-\infty}^{+\infty} \exp[\theta_i(x,k) - \theta_{i+1}(x,k) - \varphi_i(x) - \varphi_i(x+\hat{k}) + 2\pi m]^2 \right\} \right) \end{aligned} \tag{11}$$

where Z stands for the obvious normalization factor.

Correlation inequalities[11 13] may be used for the ferromagnetic coupling in (11) to give, for $g_M \geq 0$,

$$\langle A\rangle(\beta g_E, 0) \leq \langle A\rangle(\beta g_E, \beta g_M) \tag{12}$$

If we choose

$$A = \exp\{-iq[\varphi(0) - \varphi(R)]\} \tag{13}$$

then, considering (5) and (11), the left-hand side of (12) is given by

$$\langle A\rangle(\beta g_E, 0) = \frac{1}{Z_E} \operatorname{Tr}[P_{\mathbf{q}_2} \exp(-\beta g_E H_E)] \tag{14}$$

where $Z_E = \operatorname{Tr}[P_0 \exp(-\beta g_E H_E)]$.

By using the kernel (5) in (14), one finds

$$\langle A\rangle(\beta g_E, 0) = \frac{1}{Z_E}\frac{1}{(2\pi)^{|A|}}\int_0^{2\pi} d\varphi \exp\{-iq[\varphi(0)-\varphi(R)]\} \times \prod_{x,k}\left\{\left(\frac{2\pi}{\beta g_E}\right)^{1/2}\sum_{k=-\infty}^{+\infty}\exp\left[-\frac{1}{2}\frac{1}{\beta g_E}[\varphi(x)-\varphi(x+\hat{k})+2\pi m]^2\right]\right\} \tag{15}$$

But (15) is exactly the two-point function of the two-dimensional (spin) Villain model (the periodized Gaussian approximation for the plane rotator) at an effective inverse temperature $\beta^* = (\beta g_E)^{-1}$.

For the two-dimensional Villain model the following result holds

Theorem (Frohlich and Spencer[14]). There exists some finite constant β_0^* such that for $\beta^* > \beta_0^*$

$$\langle \exp\{-iq[\varphi(0)-\varphi(R)]\}\rangle(\beta^*) \geqslant \text{const}\cdot\exp\left[-\frac{q^2}{2\pi\beta'}\ln(1+|R|)\right]$$

for some positive $\beta' = \beta'(\beta^*)$, and $\beta' \to \infty$ as $\beta^* \to \infty$.

By taking logarithms in inequality (12) and using the above theorem, we obtain the following result for our model.

Theorem 2. For the two-dimensional $U(1)$ Hamiltonian gauge theory there exists a constant β_0 such that for $\beta < \beta_0$

$$V(\beta, R) \leqslant \frac{\beta''}{\beta}\frac{q^2}{2\pi}\ln(1+|R|)$$

for some positive $\beta'' = \beta''(\beta)$, and $\beta'' \to 0$ as $\beta \to 0$.

Thus, Theorem 2 gives an upper bound for $V(\beta, R)$ for β sufficiently small. As Theorem 1 provides a lower bound for all values of β, we conclude that for small β, $V(\beta, R)$ has for large R a logarithmic behavior. This is to be compared with a result due to Borgs,[4] showing that $V(\beta, R)$ is linearly increasing for large values of β and g. This configures a Kosterlitz–Thouless transition for the model we are dealing with, whose phase boundaries are sketched in Fig. 2.

In Fig. 2, region I corresponds to logarithmic confinement, as established in the present paper, while region II is a linearly confining one, as showed by Borgs.[4] Probably the phase boundary (dotted line) extends down to $g = 0$, as suggested by the work of Göpfert and Mack,[3] which

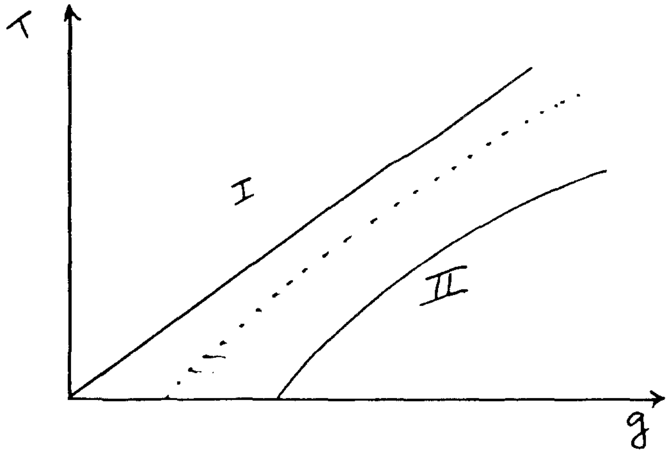

Fig. 2. A sketch of phase boundaries.

gives linear confinement in the entire $T=0$ line in the case of a *nonzero lattice spacing in time direction.* But we know of no proof of this in the continuous time model.

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